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Properties of special classes of analytic functions which are multipliers of the Cauchy-Stieltjes integrals

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Abstract

Special classes of analytic functions, denoting by K and J are considered in this paper. It is proved, that $f \in J$ if and only if $f' \in K$; $f \in J$ yildse $f \circ \phi_a \in J$, where $\phi_a(z) = z + a/1 + za$, $-1 < a < 1$; $J \subset F_0 \subset BMOA$. The classes J and K were introduced in [4].

1 Introduction

Let D denote the unit disk in the complex plane and T the unit circle. Let M be the Banach space of all complex-valued Borel measures on T with the usual variation norm.

For $\alpha \geq 0$, let F_α denote the family of the analytic functions f , for which there exists $\mu \in M$ such that

$$f(z) = \int_T \frac{1}{(1 - \bar{\xi}z)^\alpha} d\mu(\xi), \quad \alpha > 0,$$

$$f(z) = f(0) + \int_T \log \left(\frac{1}{1 - \bar{\xi}z} \right) d\mu(\xi), \quad \alpha = 0.$$

Let $\mu \in M$. In [4] were introduced the functions J_μ and K_μ by

$$J_\mu(z) = \int_T \log \left(\frac{1 - \xi z}{1 - \bar{\xi}z} \right) d\mu(\xi),$$

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$$K_\mu(z) = \int_T \left(\frac{1}{\xi - z} - \frac{\xi}{1 - \xi z} \right) d\mu(\xi) = \int_T \frac{1 - \xi^2}{(\xi - z)(1 - \xi z)} d\mu(\xi).$$

In this paper some properties of the classes J and K are given, where

$$J = \{f : f = J_\mu, \quad \mu \in M\},$$

$$K = \{f : f = K_\mu, \quad \mu \in M\}.$$

2 Main results

Let

$$\phi_a(z) = \frac{z + a}{1 + za}, \quad -1 < a < 1.$$

Lemma 1. *If $f \in K$ then $(f \circ \phi_a)' \in K$.*

Proof. Suppose that $f = K_\mu$. Then

$$\begin{aligned} (f \circ \phi_a) &= \int_T \left(\frac{1}{\xi - \frac{z+a}{1+za}} - \frac{1}{1 - \xi \frac{z+a}{1+za}} \right) d\mu(\xi) = \\ &= (1 + za) \int_T \left(\frac{1}{\xi(1 + za) - z + a} - \frac{1}{(1 + za) - \xi(z + a)} \right) d\mu(\xi) = \\ &= (1 + za) \int_T \frac{1}{1 - \xi a} \left(\frac{1}{\frac{\xi - a}{1 - \xi a} - z} - \frac{\xi}{1 - \frac{\xi - a}{1 - \xi a} z} \right) d\mu(\xi). \end{aligned}$$

Since $|a| < 1$, the function

$$\phi_a^{-1}(z) = \frac{z - a}{1 - za}$$

maps T one-to-one on T . For each Borel set $E \subset T$, let $\nu(E) = \mu(\phi_a(E))$. Then $\nu \in M$ and we can write

$$\begin{aligned} f \circ \phi_a &= (1 + za) \int_T \frac{1}{1 - \phi_a(\xi) a} \left(\frac{1}{\xi - z} - \frac{\phi_a(\xi)}{1 - \xi z} \right) d\nu(\xi) = \\ &= \frac{(1 + za)^2}{1 - a^2} \int_T \frac{1 - \xi^2}{(\xi - z)(1 - \xi z)} d\nu(\xi) = \frac{1}{\phi_a'(z)} \cdot q(z), \end{aligned}$$

where $q = K_\nu \in K$. Consequently

$$(f \circ \phi_a) \phi'_a = q(z) \in K.$$

Theorem 1.

- a) $f(z) - f(0) = J_\mu(z)$ if and only if $f'(z) = K_\mu(z)$;
- b) $J \subset F_\alpha$ for every $\alpha \geq 0$;
- c) If $f \in J$ then $f \circ \phi_a \in J$.

Proof. Suppose that $f(z) - f(0) = J_\mu(z)$.

Then f has the representation

$$f(z) - f(0) = \int_T \log \left(\frac{1 - \xi z}{1 - \bar{\xi} z} \right) d\mu(\xi),$$

with $\mu \in M$. Then

$$f'(z) = \int_T \left(\frac{-\xi}{1 - \xi z} + \frac{\xi}{1 - \bar{\xi} z} \right) d\mu(\xi) = \int_T \left(\frac{-\xi}{1 - \xi z} + \frac{1}{\xi - z} \right) d\mu(\xi) = K_\mu.$$

Conversely, suppose that $f'(z) = K_\mu(z)$, i.e

$$f'(z) = \int_T \left(\frac{1}{\xi - z} - \frac{\xi}{1 - \xi z} \right) d\mu(\xi), \quad \mu \in M.$$

Then

$$\begin{aligned} f(z) &= \int_0^z f'(t) dt + f(0) = \\ &= \int_0^z \left(\int_T \left(\frac{1}{\xi - t} - \frac{\xi}{1 - \xi t} \right) d\mu(\xi) \right) dt + f(0) = \\ &= \int_T \left(\int_0^z \left(\frac{1}{\xi - t} - \frac{\xi}{1 - \xi t} \right) dt \right) d\mu(\xi) + f(0) = \\ &= \int_T \left(\log \frac{1 - \xi z}{\xi - z} - \log \frac{1}{\xi} \right) d\mu(\xi) + f(0) = \\ &= \int_T \log \left(\frac{1 - \xi z}{1 - \bar{\xi} z} \right) d\mu(\xi) + f(0) = J_\mu(z) + f(0). \end{aligned}$$

To prove b), assume that $f(z) = J_\mu(z)$.

Then a) implies $f'(z) = K_\mu(z)$.

Since

$$f'(z) = \int_T \left(\frac{1}{\xi - z} - \frac{\xi}{1 - \xi z} \right) d\mu(\xi) = - \int_T \frac{\xi}{1 - \xi z} d\mu(\xi) - \int_T \frac{\xi}{1 - \xi z} d\mu(\bar{\xi}),$$

then $f'(z) \in F_1(z)$.

From $f'(z) \in F_1(z)$ it follows that $f(z) = J_\mu(z) \in F_0$ [3].

Theorem 3[3] implies that $F_0 \subset F_\alpha$ for $\alpha > 0$,

therefore $J_\mu(z) \in F_\alpha$ for every $\alpha \geq 0$.

We shall prove c). If $f \in J$ then $f'(z) \in K$ and by Lemma 1

$$(f' \circ \phi_a) \phi'_a \in K.$$

Consequently $(f \circ \phi_a)' \in K$ which yields $f \circ \phi_a \in J$.

Theorem 2. If $f(z) \in F_0$ then $f(z) \in BMOA$.

Proof. Let $f(z) \in F_0$. Then $f(z)$ has the representation

$$f(z) = \int_T \log \left(\frac{1}{1 - \bar{\xi} z} \right) d\mu(\xi) = \sum_{n=1}^{\infty} \frac{z^n}{n} \int_T \xi^n d\mu(\xi)$$

and

$$\left| \hat{f}(n) \right| = \left| \frac{1}{n} \int_T \xi^n d\mu(\xi) \right| \leq \frac{1}{n} \|\mu\|.$$

Now we shall make use of the fact that $BMOA \approx H^{1*}$ [1], as

$$\langle f, h \rangle = \lim_{r \rightarrow 1} \int_T \overline{f(r\xi)} h(\xi) dm(\xi), \quad f \in BMOA, \quad h \in H^1.$$

If $h \in H^1$ then

$$\begin{aligned} \left| \int_T \overline{f(r\xi)} h(\xi) dm(\xi) \right| &\leq \sum_{n=1}^{\infty} \left| \hat{f}(n) \right| r^n \left| \int_T \xi^{-n} h(\xi) dm(\xi) \right| = \\ &= \sum_{n=1}^{\infty} \left| \hat{f}(n) \right| r^n \left| \hat{h}(n) \right| \leq \|\mu\| \sum_{n=1}^{\infty} \frac{1}{n} \left| \hat{h}(n) \right| r^n. \end{aligned}$$

By Hardy's inequality

$$\sum_{n=1}^{\infty} \frac{1}{n} \left| \hat{h}(n) \right| \leq \pi \|h\|_{H^1}.$$

Therefore

$$\|f\|_{BMOA} = \sup \left\{ \lim_{r \rightarrow 1} \left| \int_T \overline{f(r\xi)} h(\xi) dm(\xi) \right|, \|h\|_{H^1} \leq 1 \right\} \leq \pi \|\mu\| < \infty.$$

Corollary. $J \subset BMOA$.

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